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Letters

# Finite-time boundedness of uncertain time-delayed neural network with Markovian jumping parameters

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ABSTRACT

The stochastic finite-time boundedness (FTB) problem is considered for a class of Markovian jumping neural networks (MJNNs) with time delay and uncertainties. By selecting the appropriate stochastic Lyapunov–Krasovskii functional, sufficient conditions of stochastic FTB of MJNNs are presented and proved. The FTB criteria are formulated in the form of linear matrix inequalities. Simulation results illustrate the effectiveness of the developed approaches.



## 1. Introduction

Since neural networks have been extensively studied in various aspects and successfully applied to various fields such as associative memories, pattern recognition, signal processing, fixed-point computations and optimization problems, it is necessary to point out that these applications are mostly built upon the stability of the equilibrium point of neural networks. For instance, when a neural network is applied as an optimization solver, the equilibrium points of the neural network characterize possible optimal solutions of the optimization problem, and starting from any initial condition, the global asymptotic stability ensures the convergence to an optimal solution. Therefore, the stability analysis is essential for the design of neural networks and then has been extensively investigated for researchers [1–4].

On another research front, time delays are frequently encountered in neural networks due to the finite switching speed of information processing and the inherent communication of neurons. The existence of time delays may cause divergence, oscillation, and even instability in dynamic systems and usually leads to unsatisfactory performances. Therefore, the problems of stability analysis of neural networks with time delays have been of considerable interest and in particular robust Lyapunov stability problem has received more consideration. For more results on this topic, we refer readers to [5–11] and the references therein. In general, Lyapunov stability is used to deal with the asymptotic pattern of system trajectories and the steady-state behaviors of control systems over an infinite-time interval. But in many practical applications, the main attentions are related to the behavior of dynamical systems over a fixed finite time interval, for instance, large values of the state are not acceptable in the presence of saturations. Therefore, we generally need to ensure that these state values are allowable by giving some initial conditions. In view of this, the finite-time stability (or short-time stability) referring to these transient performances is proposed during the 1960s [12,13]. It means that once we fix a finite-time interval, the state of a system does not exceed a certain bound during this specified time interval. Some attempts on finite-time stability can be found in [14–16] by using Lyapunov functional approach. Then, with the aid of linear matrix inequalities (LMIs) techniques, more concepts of finite-time stability have been proposed for linear continuous-time or discrete-time control system, such as finite-time boundedness (FTB), finite-time stabilization etc. Many authors have made some attempts in this regard, for instance, [17–25] and the references therein. But to the best of our knowledge, the robust FTB problems for Markovian jumping neural networks (MJNNs) with time-delays and uncertainties have not been intensively studied. This has motivated our research on this topic.

In this paper, we deal with the stochastic finite-time boundedness (FTB) problems for a class of MJNNs with time-delays and uncertain parameters. Difference with the main results in [29,30],

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the sufficient conditions of this paper are identified to guarantee solutions to stochastic boundedness via finite-time interval for such stochastic neural networks. The stochastic Lyapunov–Krasovskii functionals and the LMIs approaches are combined to investigate the problem and to derive the FTB criteria. Distinct from previous investigations, the current study focuses on the stochastic boundedness via the finite-time interval for MJNNs with constant or time-varying delays. The main advantages of the present approach include: (i) it needs no turning of parameters and/or matrices; (ii) it can be efficiently verified via solving numerically the LMI algorithms. It is noted that the results in Theorem 1 for FTB requires that the derivative of the time-varying delay be less than one, and such assumptions are often needed to deal with the stability problem of time-varying delayed neural networks in many other research papers [2,5,8]. Finally, a numerical simulation is included to illustrate the effectiveness of the developed techniques.

The rest of this paper is organized as follows. In Section 2, the problem to be studied is stated and some definitions and assumptions are presented. Based on the stochastic Lyapunov–Krasovskii stability theory, in combination with the LMIs approach, the stochastic FTB criteria for MJNNs with time-varying constant delays are then derived in Section 3, and the relevant results are also obtained for constant time delayed MJNNs. In Section 4, a numerical simulation is included to illustrate the effectiveness of the developed techniques. Finally, some conclusions are given in Section 5.

Throughout this paper, we use the following notations: $\mathfrak{R}^n$ and $\mathfrak{R}^{n\times m}$ stand for an $n$-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively; $A^{\mathrm{T}}$ and $A^{-1}$ denote the matrix transpose and matrix inverse; $diagA\quad B$ represents the block-diagonal matrix of $A$ and $B$; $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximal and minimal eigenvalue of a positive-define matrix $P$; $\|*\|$ denotes the Euclidean norm of vectors; $\mathbf{E}\{*\}$ denotes the mathematics statistical expectation of the stochastic process or vector; $L_2^n(0\quad\infty)$ is the space of $n$ dimensional square integrable function vector over $(0\quad\infty)$; $P<0$ or $P>0$ means matrix $P$ is negative-definite or positive-definite; $I$ and 0 are respectively the unit and the zero matrices with appropriate dimensions; "$*$" means the symmetric terms in a symmetric matrix.

## 2. System formulation

Given a probability space $(\Omega,F,P_r)$ where $\Omega$ is the sample space, $F$ is the algebra of events and $P_r$ is the probability measure defined on $F$, let us consider the following uncertain neural networks with Markovian jumping parameters in a fixed complete probability space $(\Omega,F,P_r)$ described by a nonlinear differential equation:

$$\begin{aligned}\dot{\delta}(t) = &-[A(r_t)+\Delta A(r_t)]\delta(t)+[B(r_t)+\Delta B(r_t)]h(t,\delta(t))\\ &+[C(r_t)+\Delta C(r_t)]h(t,\delta(t-\tau(t)))+W\end{aligned} \tag{1}$$

where $\delta(t)=[\delta_1(t)\quad\delta_2(t)\quad\cdots\quad\delta_n(t)]^T\in\mathfrak{R}^n$ is the state vector associated with $n$ neurons, $A(r_t)=diag\big(a_1(r_t)\quad a_2(r_t)\quad\cdots\quad a_n(r_t)\big)$ is the known mode-dependent diagonal matrices with positive entries $a_i(r_t)>0,\ i=1,2,\cdots,n$. The mode-dependent matrices $B(r_t)$ and $C(r_t)$ are, respectively, the connection weight matrix and the delayed connection weight matrix. $h(t,\delta(t))=[h_1(t,\delta(t))\quad h_2(t,\delta(t))\quad\cdots\quad h_n(t,\delta(t))]$ is the neuron activation function, and $W=[W_1\quad W_2\quad\cdots\quad W_n]^T$ is a constant external input vector. For presentation convenience, when $r_t{=}i,i\in M$, we denote $A(r_t)$, $\Delta A(r_t)$, $B(r_t)$, $\Delta B(r_t)$, $C(r_t)$, $\Delta C(r_t)$ as $A_i$, $\Delta A_i$, $B_i$, $\Delta B_i$, $C_i$, $\Delta C_i$. $\tau(t)$ is the time-varying delay which satisfies

$$\begin{cases}0\le\tau(t)<\overline{\tau}\\ 0\le\dot{\tau}(t)<1\end{cases} \tag{2}$$

where $\overline{\tau}$ is a constant scalar.

The uncertain parameters $\Delta A_i$, $\Delta B_i$, $\Delta C_i$ are time-varying but norm bounded, and satisfy,

$$\begin{bmatrix}\Delta A_i & \Delta B_i & \Delta C_i\end{bmatrix}=M_i\Gamma_i(t)\begin{bmatrix}N_{1i} & N_{2i} & N_{3i}\end{bmatrix} \tag{3}$$

where $M_i$, $N_{1i}$, $N_{2i}$, $N_{3i}$, are known mode-dependent matrices with appropriate dimensions and $\Gamma_i(\ t)$ is the time-varying unknown matrix function with Lebesgue norm measurable elements satisfying

$$\Gamma_i^T(t)\Gamma_i(t)\le I \tag{4}$$

in which $I$ is the identity matrix of appropriate dimension.

**Remark 1.** The uncertain parameters $\Delta A_i$, $\Delta B_i$, $\Delta C_i$ are said to be admissible if conditions (3) and (4) hold. The mode-dependent matrix $M_i$ is always chosen as a full row rank one. We always consider these uncertainties; that is because it is usually difficult to obtain the exact mathematical model of real plants due to process complexity, environmental noises, time-varying characteristics and difficulties in measuring various kinds of uncertain parameters, etc. In fact, the uncertainties described in (3) have been widely used in the schemes of stochastic robust stability of uncertain neural networks, see [8,26,27] and the references therein. We can also represent these uncertainties as state-dependent on, i.e., $\Gamma_i(\ t)=\Gamma_i(\ t,\ x(t))$, as long as $\Gamma_i^T(t,x(t))\Gamma_i(t,x(t))\le I$ is satisfied. Without these uncertainties, i.e., $\Delta A_i\equiv 0$, $\Delta B_i\equiv 0$, $\Delta C_i\equiv 0$, the time-delayed MJNNs (1) is labeled as a nominal one.

The jump parameter $r_t{=}i$ in MJNNs (1) represents a continuous-time discrete-state Markovian stochastic process taking values on a finite set $\Lambda=1,2,\cdots,N$ with transition rate matrix $P_r{=}\{P_{ij}(t),\quad i,\quad j\in\Lambda\}$, and define the following transition probability from mode $i$ at time $t$ to mode $j$ at time $t+\Delta t$ as

$$P_r\{r_{t+\Delta t}=j|r_t=i\}=\begin{cases}\pi_{ij}\Delta t+o(\Delta t), & i\ne j\\ 1+\pi_{ii}\Delta t+o(\Delta t), & i=j\end{cases} \tag{5}$$

where $\Delta t>0$; $\lim\limits_{\Delta t\to 0}o(\Delta t)/(\Delta t)=0$; $\pi_{ij}\ge 0$ is the transition probability rate from mode $i$ to mode $j$ and satisfies,

$$\sum_{j=1,j\ne i}^{N}\pi_{ij}=-\pi_{ii}\quad\text{for } i,j\in\Lambda,\ i\ne j \tag{6}$$

**Assumption 1.** The neuron state-based nonlinear function $h(t,\ \delta(t))$ in MJNNs (1) is bounded and satisfies:

$$0\le\frac{h_l(t,\xi_1)-h_l(t,\xi_2)}{\xi_1-\xi_2}\le\varsigma_l,\quad l=1,2,\cdots,n \tag{7}$$

for all $\xi_1,\ \xi_2\in\mathfrak{R}$, with $\varsigma_l$ being known real constants with $l=1,2,\cdots,n$.

Then, by using the celebrated Brouwer's fixed-pointed theorem, on can easily prove that there exists at least one equilibrium point of system (1). Let $\delta_*$ be the equilibrium point of MJNNs (1), and define $x(t){=}\delta(t)-\delta^*$. The time-delayed MJNNs (1) can be transformed as:

$$\dot{x}(t)=-A_ix(t)+B_if(t,x(t))+C_if(t,x(t-\tau(t))) \tag{8}$$

where

$$\begin{aligned}&x(t)=\begin{bmatrix}x_1(t) & x_2(t) & \cdots & x_n(t)\end{bmatrix}^T,\\ &f(t,x(t))=\begin{bmatrix}f_1(t,x_1(t)) & f_2(t,x_2(t)) & \cdots & f_n(t,x_n(t))\end{bmatrix}^T,\\ &f_l(t,x_l(t))=h_l(t,x_l(t)+\delta^*)-h_l(t,\delta^*),\quad l=1,2,\cdots,n.\end{aligned}$$

and $f_l(0){=}0$, for $l=1,2,\cdots,n$. Note that the functions $f_l(\cdot)$ satisfies the following conditions:

$$\begin{cases}|f_l(t,\xi_1)-f_l(t,\xi_2)|\le k_l|\xi_1-\xi_2|\\ |f_l(t,\xi)|\le k_l|\xi|\end{cases}. \tag{9}$$

The main aim of this paper is to develop techniques of the stochastic finite-time boundedness (FTB) problem of uncertain time-delayed MJNNs (1). The idea of this concept concerns the boundedness of the state over a finite-time interval for some given initial conditions.

**Definition 1.** The nominal time-delayed MJNNs (1) (or (8)) is said to be stochastically finite-time bounded (FTB) with respect to $(c_1 \quad c_2 \quad T)$, if

$$\mathbf{E}\|x(t_1)||^2 \le c_1 \Rightarrow \mathbf{E}\|x(t_2)\|^2 < c_2, \quad t_1 \in [-\overline{\tau} \quad 0], \quad t_2 \in [0 \quad T]. \tag{10}$$

**Definition 2.** The uncertain time-delayed MJNNs (1) (or (8)) is said to be stochastically robustly FTB with respect to $(c_1 \quad c_2 \quad T)$, if relation (10) holds for all the given uncertainties with form (3) and (4).

**Remark 2.** If we let $W=0$ in MJNNs (1), the concept of FTB reduces to finite-time stability (FTS). It is easy to see that, given our Definitions 1 and 2 of FTB, FTS can be recovered as a particular case by letting $W=0$. A system is FTB if, given a bound initial condition and a characterization of the set of admissible inputs, the system states remain below the prescribed limit for all inputs in the bound set. It should be noted that the concepts of Lyapunov stability and FTB are different. The former is largely known to the control characteristic in infinite time-interval, but the latter concerns the boundedness analysis of the controlled states within a finite time-interval. Obviously, a stochastic FTB MJNNs may not be Lyapunov stochastically stable and vice versa.

**Definition 2.** (Mao [28]) Let $V(x(t),r_t,t>0)$ be a stochastic positive functional, and define its weak infinitesimal operator as

$$\Im V(x(t),r_t=i,t) = \lim_{\Delta t \to \mathbf{0}} \frac{1}{\Delta t}[\mathbf{E}\{V(x(t+\Delta t),r_{t+\Delta t},t+\Delta t)|x(t), \\ r_t=i - V(x(t),r_t=i,t)]. \tag{11}$$

## 3. Main results

In this section, we will first study the FTB problem for nominal time-delayed MJNNs (1).

**Theorem 1.** Given a time-constant $T>0$, the nominal time-delayed MJNNs (1) is stochastically FTB with respect to $(c_1 \quad c_2 \quad T)$, if there exists a positive constant $\alpha>0$, mode-dependent symmetric positive-definite matrices $P_i>0$, mode-dependent diagonal matrices $R_i>0$, and symmetric positive-definite matrix $Q>0$, satisfying the following matrix inequalities for all $i\in\Lambda$,

$$\begin{bmatrix} \Sigma_i & P_iB_i & P_iC_i \\ * & -R_i+Q & 0 \\ * & * & -\sigma Q \end{bmatrix} < 0 \tag{12}$$

$$c_1 e^{\alpha T}\left(\lambda_{\overline{P}} + \overline{\tau}\lambda_Q \overline{k}_l\right) < \lambda_{\underline{P}} c_2 \tag{13}$$

where

$$\Sigma_i = -A_i^T P_i - P_i A_i - \sum_{j=1}^N \pi_{ij} P_j + K_l R_i K_l - \alpha P_i,$$

$$K_l = diag\{k_1 \quad k_2 \quad \cdots \quad k_n\}, \quad \sigma = \inf_{t\ge 0}(1-\dot{\tau}(t)),$$

$$\lambda_{\overline{P}} = \max_{i\in\Lambda}\lambda_{\max}(P_i), \quad \lambda_Q = \lambda_{\max}(Q), \quad \overline{k}_l = \max_l(k_l), \; \lambda_{\underline{P}} = \min_{i\in\Lambda}\lambda_{\min}(P_i).$$

**Proof.** Let the mode at time $t$ be $i$; that is $r_t = i \in \Lambda$. Take the stochastic Lyapunov–Krasovskii functional $V((t),x_t,t>0)$: $\Re^n \times \Lambda \times \Re_+ \to \Re_+$ to be

$$V\Big(x(t),r_t=i,(t) = x^T(t)P_i x(t) + \int_{t-\tau(t)}^t f^T(\xi,x(\xi))\, Qf(\xi,x(\xi))d\xi \tag{14}$$

where $P_i>0$, $Q>0$ are the given symmetric positive-definite matrices.

Along the trajectories of the nominal time-delayed MJNNs (8), the weak infinitesimal operator of the stochastic process $\{x(t),r_t=i\}|_{t\ge 0}$ is given by

$$\begin{aligned}\Im V(x(t),r_t=i,t) = &-x^T(t)\left[A_i^T P_i + P_i A_i + \sum_{j=1}^N \pi_{ij}P_j\right] \\ & x(t) + 2x^T(t)P_iB_if(t,x(t)) \\ & +2x^T(t)P_iC_if\Big(t,x(t-\tau(t))\big) + f^T(t,x(t))Qf(t,x(t)) \\ & -[1-\dot{\tau}(t)]f^T(t,x(t-\tau(t)))Qf(t,x(t-\tau(t))) \\ & +\sum_{j=1}^N \pi_{ij}P_j \int_{t-\tau(t)}^t f^T(\xi,x(\xi))Qf(\xi,x(\xi)\Big)d\xi. \end{aligned} \tag{15}$$

Let $R_i>0$ be mode-dependent diagonal matrices. We can rewrite the above equation as

$$\Im V(x(t),r_t=i,t) = \Im V(x(t),r_t=i,t) + f^T(t,x(t))R_if(t,x(t)) \\ -f^T(t,x(t))R_if(t,x(t)). \tag{16}$$

Also, it results from (9) that

$$f^T(t,x(t))R_if(t,x(t)) \le x^T(t)K_lR_iK_lx(t). \tag{17}$$

Combining (16) and (17) with (15), we can get

$$\Im V(x(t),i,t) \le \varpi^T(t)\Pi_i\varpi(t) \tag{18}$$

where $\varpi(t) = col[x(t) \quad f(t,x(t)) \quad f(t,x(t-\tau(t)))]$,

$$\Pi_i = \begin{bmatrix} -A_i^TP_i - P_iA_i - \sum_{j=1}^N \pi_{ij}P_j + K_lR_iK_l & P_iB_i & P_iC_i \\ * & -R_i+Q & 0 \\ * & * & -\sigma Q \end{bmatrix}.$$

in which $\sigma = \inf_{t\ge 0}(1-\dot{\tau}(t))$. Hence, $\Im V(x(t),i,\, t)<0$ can be held by $\Pi_i<0$.

On the other hand, it follows from inequality (13) and the required constant $\alpha>0$ that

$$\mathbf{E}[\Im V(x(t),i,t)] \le \alpha\mathbf{E}[V(x(t),i,t)]. \tag{19}$$

Multiplying (19) by $e^{-\alpha t}$, we can get

$$\mathbf{E}[\Im e^{-\alpha t}V(x(t),i,t)] \le \alpha\mathbf{E}[V(x(t),i,t)] \tag{20}$$

By integrating the above inequality from 0 to $t$, it follows that

$$e^{-\alpha t}\mathbf{E}[V(x(t),i,t)] \le \mathbf{E}[V(x_0,r_0)]. \tag{21}$$

Note that $\alpha>0$, $0\le t\le T$, we can obtain the following relation

$$\begin{aligned}\mathbf{E}[x^T(t)P_ix(t)] &< \mathbf{E}[V(x(t),i,t)] \le e^{\alpha t}\mathbf{E}[V(x_0,r_0)] \\ &= e^{\alpha t}[x^T(0)P_ix(0) + \int_{-\tau(t)}^0 f^T(\xi,x(\xi))Qf(\xi,x(\xi))d\xi] \\ &< e^{\alpha t}[\lambda_{\overline{P}}x^T(0)x(0) + \overline{\tau}\lambda_Q\overline{k}_l^2 \max_{t_1\in[-\overline{\tau} \quad 0]}(x^T(t_1)x(t_1))] \\ &< c_1e^{\alpha T}\left(\lambda_{\overline{P}} + \overline{\tau}\lambda_Q\overline{k}_l^2\right). \end{aligned} \tag{22}$$

where $\lambda_{\overline{P}} = \max_{i\in\Lambda}\lambda_{\max}(P_i), \quad \lambda_Q = \lambda_{\max}(Q), \quad \overline{k}_l = \max_l(k_l).$

Similarly, we have

$$\mathbf{E}[x^T(t)P_ix(t)] \ge \lambda_{\underline{P}}\mathbf{E}[x^T(t)x(t)] \le \lambda_{\underline{P}}\mathbf{E}\|x(t)||^2 \tag{23}$$

where $\lambda_{\underline{P}} = \min_{i\in\Lambda}\lambda_{\min}(P_i)$.

Then we can get

$$\mathbf{E}\|x(t)||^2 < \frac{c_1e^{\alpha T}\left(\lambda_{\overline{P}} + \overline{\tau}\lambda_Q\overline{k}_l^2\right)}{\lambda_{\underline{P}}} < c_2 \tag{24}$$

It implies by condition (14) that for $\forall t \in [0 \quad T]$, $\mathbf{E}[\|x(t)\|^2] < c_2$. This completes the Proof.

Besides the time-delay, parameter uncertainties are still the inherent features of many physical processes and often encountered in engineering systems, their presences must be considered. Before proceeding with this kind of time-delayed MJNNs with uncertainties, the following Lemmas are needed.

**Lemma 1.** [29] Let $T$, $M$, $F$ and $N$ be real matrices of appropriate dimension with $F^TF \le I$, then for a positive scalar $\alpha > 0$, it holds

$$T + MFN + N^TF^TM^T \le T + \alpha^{-1}MM^T + \alpha N^TN \quad (25)$$

**Theorem 2.** Given a time-constant $T > 0$, the uncertain time-delayed MJNNs (1) is stochastically robustly *FTB* with respect to $(c_1 \quad c_2 \quad T)$, if there exists a positive constant $\alpha > 0$, a mode-dependent symmetric positive-definite matrix $P_i > 0$, a mode-dependent diagonal matrix $R_i > 0$, symmetric positive-definite matrix $Q > 0$, and mode-dependent scalars $\beta_i > 0$, satisfying relation (13) and the following matrix inequality for all $i \in \Lambda$,

$$\begin{bmatrix} \Sigma_{2i} & P_iB_i - \beta_iN_{1i}^TN_{2i} & P_iC_i - \beta_iN_{1i}^TN_{3i} & P_iM_i \\ * & -R_i + Q + \beta_iN_{2i}^TN_{2i} & \beta_iN_{2i}^TN_{3i} & 0 \\ * & * & -\sigma Q + \beta_iN_{3i}^TN_{3i} & 0 \\ * & * & * & -\beta_iI \end{bmatrix} \quad (26)$$

where

$$\Sigma_i = -A_i^TP_i - P_iA_i - \sum_{j=1}^{N} \pi_{ij}P_j + K_lR_iK_l + \beta_iN_{1i}^TN_{1i} - \alpha P_i,$$

$$K_l = diag\{k_1 \quad k_2 \quad \cdots \quad k_n\}, \quad \sigma = \inf_{t \ge 0}(1 - \dot{\tau}(t))\Big).$$

**Proof.** Take the same stochastic Lyapunov–Krasovskii functional as in the proof of Theorem 1, and along the trajectories of the uncertain time-delayed MJNNs (8), the weak infinitesimal operator of the stochastic process $\{x(t), r_t = i\}|_{t \ge 0}$is given by

$$\begin{aligned} \Im V\Big(x(t), r_t = i, (t) &= -x^T(t)\left[(A_i + \Delta A_i)^TP_i + P_i(A_i + \Delta A_i) + \sum_{j=1}^{N}\pi_{ij}P_j\right]x(t) \\ &+ 2x^T(t)P_i(B_i + \Delta B_i)f(t, x(t)) + 2x^T(t)P_i(C_i + \Delta C_i)f(t, x(t - \tau(t))) \\ &+ f^T(t, x(t))Qf(t, x(t)) - [1 - \dot{\tau}(t)]f^T(t, x(t - \tau(t)))Qf(t, x(t - \tau(t))) \\ &+ \sum_{j=1}^{N}\pi_{ij}P_j\int_{t-\tau(t)}^{t} f^T(\xi, x(\xi))Qf(\xi, x(\xi))d\xi. \end{aligned} \quad (27)$$

Then, we can get

$$\Im V(x(t), i, t) \le \varpi^T(t)\Phi_i\varpi(t) \quad (28)$$

where

$$\Phi_i = \begin{bmatrix} \Phi_{1i} & P_i(B_i + \Delta B_i) & P_i(C_i + \Delta C_i) \\ * & -R_i + Q & 0 \\ * & * & -\sigma Q \end{bmatrix},$$

in which $\Phi_{1i} = -(A_i + \Delta A_i)^TP_i - P_i(A_i + \Delta A_i) - \sum_{j=1}^{N}\pi_{ij}P_j + K_lR_iK_l$.

Hence, $\Im V(x(t), i,\ t) < 0$ can be held by $\Phi_i < 0$.

In order to dealt with the uncertainties described as the form in (3) and (4), we can use the following approach:

$$\Phi_i = \Pi_i + \Delta\Pi_i < 0,$$

where

$$\Delta\Pi_i = \begin{bmatrix} -\Delta A_i^TP_i - P_i\Delta A_i & P_i\Delta B_i & P_i\Delta C_i \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}.$$

Then $\Phi_i = \Pi_i + \Delta\Pi_i < 0$ is equivalent to

$$\Pi_i + \Delta\Pi_i = L_{11}\Gamma_i(t)L_{12} + L_{12}^T\Gamma_i^T(t)L_{11}^T < 0$$

where $L_{11} = col[P_iM_i \quad 0 \quad 0]$, $L_{12} = [-N_{1i} \quad N_{2i} \quad N_{3i}]$.

From Lemma 1, the above matrix inequality holds for all $\Gamma_i(\ t)$ satisfying $\|\Gamma_i(t)\| \le 1$ if and only if there exists a series of mode-dependent positive scalars $\beta_i > 0$, such that

$$\Pi_i + \beta_i^{-1}L_{11}L_{11}^T + \beta_iL_{12}^TL_{12} < 0$$

that is,

$$\Phi_i = \begin{bmatrix} \Phi_{2i} & P_iB_i - \beta_iN_{1i}^TN_{2i} & P_iC_i - \beta_iN_{1i}^TN_{3i} & P_iM_i \\ * & -R_i + Q + \beta_iN_{2i}^TN_{2i} & \beta_iN_{2i}^TN_{3i} & 0 \\ * & * & -\sigma Q + \beta_iN_{3i}^TN_{3i} & 0 \\ * & * & * & -\beta_iI \end{bmatrix} < 0 \quad (29)$$

where $\Phi_{2i} = -A_i^TP_i - P_iA_i - \sum_{j=1}^{N}\pi_{ij}P_j + K_lR_iK_l + \beta_iN_{1i}^TN_{1i}$.

Following the similar proof in Theorem 1, we can easily get the main results of . This completes the Proof. □

When there are difficulties in solving (13), we can transform (13) into the following conditions:

$$I < P_i < \sigma_1 I \quad (30)$$

$$0 < Q < \sigma_2 I \quad (31)$$

$$c_1\left(\sigma_1 + \overline{\tau}\sigma_2\overline{k}_l\right) < e^{-\alpha T}c_2 \quad (32)$$

with $\sigma_1 > 0$, $\sigma_2 > 0$.

For the uncertain MJNNs (1) with constant time-delays, which can be described as $\tau(t) = \tau > 0$, then in terms of LMIs, we obtain the following sufficient condition for the stochastic FTB.

**Theorem 3.** Given a time-constant $T > 0$, the uncertain constant time-delayed MJNNs (1) is stochastically robustly FTB with respect to $(c_1 \quad c_2 \quad T)$, if there exists a positive constant $\alpha > 0$, a mode-dependent symmetric positive-definite matrix $P_i > 0$, a mode-dependent diagonal matrix $R_i > 0$, symmetric positive-definite matrix $Q > 0$, and mode-dependent scalars $\beta_i > 0$, satisfying relations (30)–(32) and the following matrix inequality for all $i \in \Lambda$,

$$\begin{bmatrix} \Sigma_{2i} & P_iB_i - \beta_iN_{1i}^TN_{2i} & P_iC_i - \beta_iN_{1i}^TN_{3i} & P_iM_i \\ * & -R_i + Q + \beta_iN_{2i}^TN_{2i} & \beta_iN_{2i}^TN_{3i} & 0 \\ * & * & -\sigma Q + \beta_iN_{3i}^TN_{3i} & 0 \\ * & * & * & -\beta_iI \end{bmatrix}. \quad (33)$$

**Remark 3.** Theorems 1 and 2 have presented the sufficient condition of analyzing the FTB of MJNNs (1). The coupled LMIs (12) (or (26)) and LMIs (30)–(32) are respect to $P_i$, $R_i$, $Q$, $\beta_i$, $c_1$, $c_2$, $\sigma_1$, $\sigma_2$, $T$, $\overline{\tau}$ and $\alpha$. For given scalars $c_1$, $T$ and $\alpha$, we can take $c_2$ as the optimal value and optimize over value $c_2$. Similarly, we can fix $c_2$ and look for the maximum admissible $c_1$ guaranteeing the FTB of MJNNs in (1). By using the MATLAB LMIs Toolbox, it is straightforward to check the feasibility of Theorems 1 and 2. In order to illustrate the effectiveness of the developed techniques, we will give two numerical examples about dynamic MJNNs (1) in Section 4.

**Remark 4.** For the infeasible frequency methods for stochastic dynamic MJNNs (1), finite-time stability or boundedness can be considered as the extension concept of peak value or energy value performance of the dynamical systems. Following the same lines of the proof of Theorem 1, we can also get the sufficient FTB condition for the uncertain MJNNs case. It should be observed out that the novelty of the results in this paper pays more attention to the nonlinear parameters and time-varying delays appearing in the MJNNs and the relevant stability analysis with respect to the finite-time interval. Without consider the jumping parameters, and the time-interval turns to infinite-time, the main results can be reduced to [2,4,5,7] and the references therein.

## 4. Numerical examples

**Example 1.** Consider a class of MJNNs (1) with two operation modes described as follows:

$$A_1=\begin{bmatrix}2&0\\0&1\end{bmatrix},\quad B_1=\begin{bmatrix}0.5&1\\-0.2&0.5\end{bmatrix},\quad C_1=\begin{bmatrix}0.9&0.1\\-0.1&0.1\end{bmatrix},$$
$$A_2=\begin{bmatrix}3&0\\0&2\end{bmatrix},\quad B_2=\begin{bmatrix}1.1&1\\-0.2&0.1\end{bmatrix},\quad C_2=\begin{bmatrix}0.3&-0.8\\0.1&0.2\end{bmatrix},\quad K_l=I_2.$$

The mode switching is governed by a Markov chain that has the following transition rate matrix:

$$\Pi=\begin{bmatrix}-0.5&0.5\\0.3&-0.3\end{bmatrix}.$$

In this note, we choose the initial values for $c_1=0.25$, $T=2$, $\alpha=1$ and describe the time-delays as $\tau(t)=0.2\cdot|\cos t|$. Since $0\le|\cos t|\le 1$, we can get that $\overline{\tau}(t)=0.2$. From (3), it follows that $\sigma=\inf_{t\ge 0}(1-\dot{\tau}(t))=0.8$. By applying Theorem 1 and optimize over value $c_2$, we find tine-delayed MJNNs (1) is stochastically FTB with respect to $(c_1\quad c_2\quad T)$ with the minimal $c_2=5.4296$. The solution of LMIs (12) and (30)–(32) is given by:

$$P_1=\begin{bmatrix}1.0013&-0.0004\\-0.0004&1.0118\end{bmatrix},\quad P_2=\begin{bmatrix}1.5736&0.8322\\0.8322&2.2097\end{bmatrix},$$
$$Q=\begin{bmatrix}0.3853&-0.0629\\-0.0629&0.7538\end{bmatrix},$$
$$R_1=\begin{bmatrix}1.0118&0\\0&1.5736\end{bmatrix},\quad R_2=\begin{bmatrix}0.8322&0\\0&2.2097\end{bmatrix}.$$

**Example 2.** Consider two operation modes time-delayed MJNNs (1) with uncertain parameters described as follows:

$$A_1=\begin{bmatrix}3&0\\0&3\end{bmatrix},\quad B_1=\begin{bmatrix}0.3&0.2\\-0.2&0.3\end{bmatrix},\quad C_1=\begin{bmatrix}0.2&0.1\\-0.1&0.1\end{bmatrix},$$
$$A_2=\begin{bmatrix}2&0\\0&2\end{bmatrix},\quad B_2=\begin{bmatrix}-0.4&0.3\\0.5&0.1\end{bmatrix},\quad C_2=\begin{bmatrix}-0.3&0.2\\0.1&0.2\end{bmatrix},$$
$$M_1=\begin{bmatrix}-0.2&0.1\\0.1&0.1\end{bmatrix},\quad M_2=\begin{bmatrix}-0.1&0.2\\-0.1&-0.2\end{bmatrix},\quad N_{11}=\begin{bmatrix}0.2&0.5\\-0.1&0.3\end{bmatrix},$$
$$N_{12}=\begin{bmatrix}0.3&-0.1\\0.2&-0.1\end{bmatrix},\quad N_{21}=\begin{bmatrix}0.4&0.2\\-0.1&0.2\end{bmatrix},\quad N_{22}=\begin{bmatrix}0.1&-0.3\\0.4&-0.2\end{bmatrix},$$
$$N_{31}=\begin{bmatrix}0.2&-0.1\\-0.2&0.3\end{bmatrix},\quad N_{23}=\begin{bmatrix}-0.1&0.1\\0.2&-0.5\end{bmatrix},\quad K_l=I_2.$$

With the same mode switching rates, initial values and time-delays, we find tine-delayed MJNNs (1) is stochastically FTB with respect to $(c_1\quad c_2\quad T)$ with the minimal $c_2=3.0093$. The solution of (26) and (30)–(32) is given by:

$$P_1=\begin{bmatrix}1.4801&-0.1043\\-0.1043&1.4768\end{bmatrix},\quad P_2=\begin{bmatrix}1.2959&0.2990\\0.2990&1.3031\end{bmatrix},$$
$$Q=\begin{bmatrix}0.0990&0.0476\\0.0476&0.0886\end{bmatrix},$$
$$R_1=\begin{bmatrix}1.4768&0\\0&1.2959\end{bmatrix},\quad R_2=\begin{bmatrix}0.2990&0\\0&1.3031\end{bmatrix}.$$

## 5. Conclusions

In this paper, we have discussed the stochastically FTB problem for MJNNs with both time-delays and uncertain parameters. By employing a Lyapunov–Krasovskii functional, the addressed FTB analysis problem can also be converted into a convex optimization problem, and a LMI approach has been utilized to establish the sufficient conditions for the robust FTB for the MJNNs, with or without parameter uncertainties. These conditions can be readily checked by utilizing the Matlab LMI toolbox. A numerical example has been provided to demonstrate the usefulness of the proposed methods.

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